

Multivalued stochastic partial differential-integral equations via backward doubly stochastic differential equations driven by a Lévy process *

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Abstract

In this paper, we deal with a class of backward doubly stochastic differential equations (BDSDEs, in short) involving subdifferential operator of a convex function and driven by Teugels martingales associated with a Lévy process. We show the existence and uniqueness result by means of Yosida approximation. As an application, we give the existence of stochastic viscosity solution for a class of multivalued stochastic partial differential-integral equations (MSPIDEs, in short).

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1 Introduction

Backward stochastic differential equations (BSDEs, in short) related to a multivalued maximal monotone operator defined by the subdifferential of a convex function have first been introduced by Gegout-Petit and Pardoux [14]. Further, Pardoux and Răşcanu [26] proved the existence and uniqueness of the solution of BSDEs, on a random (possibly infinite) time interval, involving a subdifferential operator in order to give a probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Following, Ouknine [23], N'zi and Ouknine [19, 20], Bahlali et al. [3, 4] discussed this type of BSDEs driven by a Brownian motion or the combination of a Brownian motion and an independent Poisson point process under the conditions of Lipschitz, locally Lipschitz or some monotone conditions on the coefficients.

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Recently, a new class of BSDEs, named backward doubly stochastic differential equations (BDSDEs, in short) involving a standard forward stochastic integral and a backward stochastic integral has been introduced by Pardoux and Peng [25] in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs, in short). Following it, Matoussi and Scheutzow [18], Bally and Matoussi [5], Zhang and Zhao [31], Aman and Mrhardy[1] and Boufoussi et al. [6, 7] studied this kind of BDSDEs from different aspects.

The main tool in the theory of BSDEs is the martingale representation theorem for a martingale which is adapted to the filtration of a Brownian motion or a Poisson point process (Pardoux and Peng [24], Tang and Li [30]). Recently, Nualart and Schoutens [21] gave a martingale representation theorem associated with a Lévy process. This class of Lévy processes includes Brownian motion, Poisson process, Gamma process, negative binomial process and Meixner process as special cases. Based on [21], they showed the existence and uniqueness of the solution for BSDEs driven by Teugels martingales associated with a Lévy process in [22]. These results were important from a pure mathematical point of view as well as from application point of view in the world of finance. Specifically, they could be used for the purpose of option pricing in a Lévy market and related partial differential equation which provided an analogue of the famous Black-Scholes formula. Motivated by [25] and [22], Ren et al. [29] considered a class of BDSDEs driven by Teugels martingales and an independent Brownian motion, obtained the existence and uniqueness of solutions to these equations, which allowed to give a probabilistic interpretation for the solution to a class of stochastic partial differential-integral equations (SPDIEs, in short). Very recently, Ren and Fan [28] derived the existence and uniqueness of the solution for BSDEs driven by a Lévy process involving a subdifferential operator and gave a probabilistic interpretation for the solutions of a class of partial differential-integral inclusions (PDIIs, in short).

Motivated by the above works, the first aim of this paper is to derive existence and uniqueness result to the following BDSDE involving subdifferential operator of a convex function and driven by Teugels martingales associated with a Lévy process

$$\begin{cases} dY_t + f(t, Y_t, Z_t) dt + g(t, Y_t, Z_t) dB_t \in \partial\varphi(Y_t) dt + \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, & 0 \leq t \leq T, \\ Y_T = \xi, \end{cases} \quad (1.1)$$

where $\partial\varphi$ is a subdifferential operators. The integral with respect to $\{B_t\}$ is a backward Kunita-Itô integral (see Kunita [16]) and this one with respect to $\{H_t^{(i)}\}_{i \geq 1}$ is a standard forward Itô integral (see Gong [15]). Our method is based on the Yosida approximation.

On the other hand, since the pioneering paper due to Buckdahn and Ma [9],[10],[11], the notion of stochastic viscosity solution has been intensely studied in the last ten year. Among others, we can cite the work of Boufoussi et al. [6], [7], Aman and Mrhardy [1], Aman and Ren [2] and Ren et al. [27], etc. In all these different works, authors have set existence results to stochastic viscosity solution of several types of SPDE. The tool is entirely probabilistic and used the connection between these SPDE and associated BDSDEs. Following this way, the second goal in this paper is to give stochastic viscosity solution for

multivalued stochastic partial differential-integral equations (MSPDIEs, in short)

$$\begin{cases} \left(\frac{\partial u}{\partial t}(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), (u_k^1(t, x))_{k=1}^\infty) + g(t, x, u(t, x))\dot{B}_t \right) \in \partial\phi(x), & 0 < t < T, x \in \mathbb{R}^d \\ u(T, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

where \mathcal{L} is the second-order differential integral operator of the diffusion process given by

$$\begin{aligned} \mathcal{L}\phi(t, x) &= m_1 \sum_{i=1}^d \sigma_i(x) \frac{\partial \phi}{\partial x_i}(t, x) + \frac{1}{2} \sum_{i=1}^d \sigma_i^2(x) \frac{\partial^2 \phi}{\partial x_i^2}(t, x) \\ &\quad + \int_{\mathbb{R}} [\phi(t, x + \sigma(x)y) - \phi(t, x) - \langle \nabla \phi(t, x), \sigma(x)y \rangle] \nu(dy), \end{aligned} \quad (1.3)$$

and

$$\phi_k^1(t, x) = \int_{\mathbb{R}} (\phi(t, x + \sigma(x)y) - \phi(t, x)) p_k(y) \nu(dy),$$

with σ a \mathbb{R}^d -valued function, coefficient of SDE driven by the Lévy process L , $m_1 = \mathbb{E}(L_1)$ and p_k precise later. Our method is also fully probabilistic and uses connection between MSPDIE (1.1) and BDSDE (1.1) in Markovian framework.

The paper is organized as follows. In Section 2, we introduce some preliminaries and notations. Section 3 is devoted to the existence and uniqueness result for BDSDEs involving subdifferential operator of a convex function and driven by a Lévy process. Finally, in section 4 we derive a probabilistic representation (in stochastic viscosity sense) for the solution of a class of MSPDIEs via BDSDEs proposed in Section 3.

2 Preliminaries and notations

Let $T > 0$ be a fixed terminal time and $\{B_t : t \in [0, T]\}$ be a standard \mathbb{R} -valued Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us also consider $\{L_t : t \in [0, T]\}$, a \mathbb{R} -valued Lévy process corresponding to a standard Lévy measure ν , defined on a complete probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, with the following characteristic function:

$$\mathbb{E}(e^{iuL_t}) = \exp \left[iaut - \frac{1}{2} \kappa^2 u^2 t + t \int_{\mathbb{R}} (e^{iux} - 1 - iux 1_{\{|x| < 1\}}) \nu(dx) \right],$$

where $a \in \mathbb{R}$, $\kappa \geq 0$. Moreover, the Lévy measure ν satisfies the following conditions:

1. $\int_{\mathbb{R}} (1 \wedge y^2) \nu(dy) < \infty$;
2. $\int_{]-\varepsilon, \varepsilon[} e^{\lambda|y|} \nu(dy) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$;

which provides that L_t has moments of all orders, i.e. $\int_{-\infty}^{+\infty} |x|^i \nu(dx) < \infty$, $\forall i \geq 2$.

Let consider the product space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, defined by

$$\bar{\Omega} = \Omega \times \Omega'; \quad \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}'; \quad \bar{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'.$$

Further, random variables $\xi(\omega)$, $\omega \in \Omega$ and $\zeta(\omega')$, $\omega' \in \Omega'$ can be considered as random variables on $\bar{\Omega}$ via the following identifications:

$$\xi(\omega, \omega') = \xi(\omega); \quad \zeta(\omega, \omega') = \zeta(\omega').$$

In this fact, the processes B and L are assumed independent. Next, denoting by \mathcal{N} the totality of $\bar{\mathbb{P}}$ -null sets of $\bar{\mathcal{F}}$, and for each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_{t,T}^B \otimes \mathcal{F}_t^L \vee \mathcal{N},$$

where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s : s \leq r \leq t\}$ and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. Since $\{\mathcal{F}_{t,T}^B\}_{t \geq 0}$ is decreasing and $\{\mathcal{F}_t^L\}_{t \geq 0}$ is increasing, the object $\{\mathcal{F}_t\}_{t \geq 0}$ is neither increasing nor decreasing. Thus it does not a filtration.

We denote by $(H^{(i)})_{i \geq 1}$ the linear combination of so-called Teugels martingale $Y_t^{(i)}$ associated with the Lévy process $\{L_t : t \in [0, T]\}$. More precisely

$$H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + \cdots + c_{i,1}Y_t^{(1)},$$

where for all $i \geq 1$, $Y_t^{(i)} = L_t^{(i)} - E[L_t^{(i)}] = L_t^{(i)} - tE[L_1^{(i)}]$. For each $t \in [0, T]$, $L_t^{(i)}$ is a power-jump processes defined as follows: $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0 < s \leq t} (\triangle L_s)^i$ for $i \geq 2$. It was shown in Nualart and Schoutens [21] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \kappa^2\delta_0(dx)$:

$$\int_{\mathbb{R}} q_n(x)q_m(x)\mu(dx) = 0 \text{ if } n \neq m \text{ and } \int_{\mathbb{R}} q_n^2(x)\mu(dx) = 1.$$

We set

$$p_k(x) = xq_{k-1}(x).$$

The martingales $(H^{(i)})_{i \geq 1}$ can be chosen to be pairwise strongly orthonormal martingales, i.e. $[H^{(i)}, H^{(j)}] = 0, i \neq j$, and $\{[H^{(i)}, H^{(i)}]_t - t\}_{t \geq 0}$ are uniformly integrable martingales with initial value 0 and $\langle H^{(i)}, H^{(j)} \rangle_t = \delta_{ij}t$.

Remark 2.1. The case of $\nu = 0$ corresponds to the classic Brownian case and all non-zero degree polynomials $q_i(x)$ will vanish, giving $H_t^{(i)} = 0, i = 2, 3, \dots$, i.e. all power jump processes of order strictly greater than one will be equal to zero. If ν only has mass at 1, we have the Poisson case; here also $H_t^{(i)} = 0, i = 2, 3, \dots$, i.e. all power jump processes will be the same, and equal to the original Poisson process. Both cases are degenerate in this Lévy framework.

Let us introduce the following appropriate spaces:

- $\ell^2 = \left\{ x = (x_n)_{n \geq 1}; \|x\|_{\ell^2} = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} < \infty \right\}$.
- \mathcal{H}^2 the subspace of the \mathcal{F}_t -measurable and \mathbb{R} -valued processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{\mathcal{H}^2}^2 = \mathbb{E} \int_0^T |Y_t|^2 dt < +\infty.$$

- S^2 the subspace of the \mathbb{R} -valued, \mathcal{F}_t -measurable, right continuous left limited (rcll, in short) processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{S^2}^2 = \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) < +\infty.$$

- $\mathcal{P}^2(l^2)$ the space of jointly predictable processes $(Z)_{t \in [0, T]}$ taking values in ℓ^2 such that

$$\|Z\|_{\mathcal{P}^2(l^2)}^2 = \mathbb{E} \int_0^T \|Z_s\|_{\ell^2}^2 ds = \sum_{i=1}^{\infty} \mathbb{E} \int_0^T |Z_s^{(i)}|^2 ds < \infty.$$

Now, we give the following assumptions:

- (H1) The coefficients $f : [0, T] \times \Omega \times \mathbb{R} \times \ell^2 \rightarrow \mathbb{R}$ and $g : [0, T] \times \Omega \times \mathbb{R} \times \ell^2 \rightarrow \mathbb{R}$ satisfy, for all $t \in [0, T], y \in \mathbb{R}$ and $z \in \ell^2$,

- (i) $f(t, \cdot, y, z)$ and $g(t, \cdot, y, z)$ are \mathcal{F}_t -measurable,
- (ii) $f(\cdot, 0, 0), g(\cdot, 0, 0) \in \mathcal{H}^2$;

- (H2) There exists some constants $C > 0$ and $0 < \alpha < 1$ such that for every $(t, \omega) \in [0, T] \times \Omega, (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \ell^2$

$$|f(t, \omega, y_1, z_1) - f(t, \omega, y_2, z_2)|^2 \leq C (|y_1 - y_2|^2 + \|z_1 - z_2\|_{\ell^2}^2),$$

$$|g(t, \omega, y_1, z_1) - g(t, \omega, y_2, z_2)|^2 \leq C |y_1 - y_2|^2 + \alpha \|z_1 - z_2\|_{\ell^2}^2;$$

- (H3) Let $\varphi : \mathbb{R} \rightarrow (-\infty, +\infty]$ be a proper lower semi continuous convex function satisfying $\varphi(y) \geq \varphi(0) = 0$;

- (H4) The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ satisfies

$$\mathbb{E} (|\xi|^2 + \varphi(\xi)) < \infty.$$

Define:

$$\text{Dom}(\varphi) = \{u \in \mathbb{R} : \varphi(u) < +\infty\},$$

$$\partial\varphi(u) = \{u^* \in \mathbb{R} : \langle u^*, v - u \rangle + \varphi(u) \leq \varphi(v), \text{ for all } v \in \mathbb{R}\},$$

$$\text{Dom}(\partial\varphi) = \{u \in \mathbb{R} : \partial\varphi(u) \neq \emptyset\},$$

$$\text{Gr}(\partial\varphi) = \{(u, u^*) \in \mathbb{R}^2 : u \in \text{Dom}(\partial\varphi), u^* \in \partial\varphi(u)\}.$$

Now, we introduce a multi-valued maximal monotone operator on \mathbb{R} defined by the subdifferential of the above function φ .

For all $x \in \mathbb{R}$, define

$$\varphi_\varepsilon(x) = \min_y \left(\frac{1}{2} |x - y|^2 + \varepsilon \varphi(y) \right) = \frac{1}{2} |x - J_\varepsilon(x)|^2 + \varepsilon \varphi(J_\varepsilon(x)),$$

where $J_\varepsilon(x) = (I + \varepsilon \partial\varphi)^{-1}(x)$ is called the resolvent of the monotone operator $A = \partial\varphi$. Then, we have the following Proposition which appeared in Brezis [8].

Proposition 2.2. (1) $\varphi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function with Lipschitz continuous derivatives;

$$(2) \quad \forall x \in \mathbb{R}, \frac{1}{\varepsilon} D\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \partial\varphi_\varepsilon(x) = \frac{1}{\varepsilon} (x - J_\varepsilon(x)) \in \partial\varphi(J_\varepsilon(x));$$

$$(3) \quad \forall x, y \in \mathbb{R}, |J_\varepsilon(x) - J_\varepsilon(y)| \leq |x - y|;$$

$$(4) \quad \forall x \in \mathbb{R}, 0 \leq \varphi_\varepsilon(x) \leq \langle D\varphi_\varepsilon(x), x \rangle;$$

$$(5) \quad \forall x, y \in \mathbb{R} \text{ and } \varepsilon, \delta > 0, \langle \frac{1}{\varepsilon} D\varphi_\varepsilon(x) - \frac{1}{\delta} D\varphi_\delta(y), x - y \rangle \geq -\left(\frac{1}{\varepsilon} + \frac{1}{\delta}\right) |D\varphi_\varepsilon(x)| |D\varphi_\delta(y)|.$$

We first give the definition of BDSDEs involving subdifferential operator of a convex function and driven by Lévy process.

Definition 2.3. We call solution of BDSDE (ξ, f, g, φ) a triple of (Y, U, Z) of progressively measurable processes such that

1. $(Y, Z) \in S^2 \times \mathcal{P}^2(l^2)$, $U \in \mathcal{H}^2$;
2. $(Y_t, U_t) \in \partial\varphi$, $d\mathbb{P} \otimes dt$ -a.e. on $[0, T]$;
3. $Y_t + \int_t^T U_s ds = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}, 0 \leq t \leq T$.

3 Existence and uniqueness result for BDSDE driven by Lévy process

The first result of the paper is the following theorem:

Theorem 3.1. Assume the assumptions of (H1)–(H4) hold. Then, the BDSDE (ξ, f, g, φ) has a unique solution.

For the prove of this theorem, let us consider the following BDSDEs:

$$\begin{aligned} Y_t^\varepsilon + \frac{1}{\varepsilon} \int_t^T D\varphi_\varepsilon(Y_s^\varepsilon) ds &= \xi + \int_t^T f(s, Y_s^\varepsilon, Z_s^\varepsilon) ds + \int_t^T g(s, Y_s^\varepsilon, Z_s^\varepsilon) dB_s \\ &\quad - \sum_{i=1}^{\infty} \int_t^T Z_s^{\varepsilon, (i)} dH_s^{(i)}, \quad 0 \leq t \leq T, \end{aligned} \quad (3.1)$$

where φ_ε is the Yosida approximation of the operator $A = \partial\varphi$. Since $\frac{1}{\varepsilon} D\varphi_\varepsilon(Y_s^\varepsilon)$ is Lipschitz continuous, it is known from a recent result of Ren et al. [29], that Eq. (3.1) has a unique solution $(Y^\varepsilon, Z^\varepsilon) \in S^2 \times \mathcal{P}^2(l^2)$.

Setting $U_t^\varepsilon = \frac{1}{\varepsilon} D\varphi_\varepsilon(Y_t^\varepsilon)$, $0 \leq t \leq T$, our aim is to prove that the sequence $(Y^\varepsilon, U^\varepsilon, Z^\varepsilon)$ converges to a sequence (Y, U, Z) which is the desired solution of the BDSDEs.

In the sequel, $C > 0$ is a constant which can change its value from line to line. Firstly, we give a prior estimates on the solution.

Lemma 3.2. Assume the assumptions of (H1)–(H4) hold. Then, there exists a constant $C_1 > 0$ such that for all $\varepsilon > 0$

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2 + \int_0^T \|Z_s^\varepsilon\|_{\ell^2}^2 ds \right) \leq C_1.$$

Proof. Applying the Itô formula to $|Y_t^\varepsilon|^2$ yields that

$$\begin{aligned} |Y_t^\varepsilon|^2 + \frac{2}{\varepsilon} \int_t^T Y_s^\varepsilon D\phi_\varepsilon(Y_s^\varepsilon) ds &= |\xi|^2 + 2 \int_t^T Y_{s-}^\varepsilon f(s, Y_s^\varepsilon, Z_s^\varepsilon) ds + 2 \int_t^T Y_{s-}^\varepsilon g(s, Y_s^\varepsilon, Z_s^\varepsilon) dB_s \\ &\quad + \int_t^T |g(s, Y_s^\varepsilon, Z_s^\varepsilon)|^2 ds - \sum_{i=1}^{\infty} \int_t^T |Z_s^{\varepsilon, (i)}|^2 d[H^{(i)}, H^{(i)}]_s \\ &\quad - 2 \sum_{i=1}^{\infty} \int_t^T Y_{s-}^\varepsilon Z_s^{\varepsilon, (i)} dH_s^{(i)}. \end{aligned} \quad (3.2)$$

Noting that the fact $Y_s^\varepsilon D\phi_\varepsilon(Y_s^\varepsilon) \geq 0$ and taking expectation on the both sides, we obtain

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon|^2 + \mathbb{E} \int_t^T \|Z_s^\varepsilon\|_{\ell^2}^2 ds &\leq \mathbb{E}|\xi|^2 + 2\mathbb{E} \int_t^T Y_{s-}^\varepsilon f(s, Y_s^\varepsilon, Z_s^\varepsilon) ds \\ &\quad + \mathbb{E} \int_t^T |g(s, Y_s^\varepsilon, Z_s^\varepsilon)|^2 ds. \end{aligned} \quad (3.3)$$

Using the elementary inequality $2ab \leq \beta^2 a^2 + \frac{b^2}{\beta^2}$ for all $a, b \geq 0$ and (H2), we get

$$\begin{aligned} 2yf(s, y, z) &= 2y(f(s, y, z) - f(s, 0, 0)) + 2yf(s, 0, 0) \\ &\leq \frac{1}{M}|y|^2 + MC|y|^2 + MC\|z\|_{\ell^2}^2 + |y|^2 + |f(s, 0, 0)|^2 \\ &\leq \left(1 + \frac{1}{M} + MC\right)|y|^2 + |f(s, 0, 0)|^2 + MC\|z\|_{\ell^2}^2, \end{aligned}$$

$$\begin{aligned} |g(s, y, z)|^2 &= |g(s, y, z) - g(s, 0, 0) + g(s, 0, 0)|^2 \\ &\leq \left(1 + \frac{1}{\beta}\right)|g(s, y, z) - g(s, 0, 0)|^2 + (1 + \beta)|g(s, 0, 0)|^2 \\ &\leq \left(1 + \frac{1}{\beta}\right)C|y|^2 + (1 + \beta)|g(s, 0, 0)|^2 + \alpha \left(1 + \frac{1}{\beta}\right)\|z\|_{\ell^2}^2. \end{aligned}$$

Choosing $M = \frac{1-\alpha}{2C}$, $\beta = \frac{3\alpha}{1-\alpha}$, it follows from (3.3) that

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon|^2 + \frac{1-\alpha}{6}\mathbb{E} \int_t^T \|Z_s^\varepsilon\|_{\ell^2}^2 ds \\ \leq C\mathbb{E} \left(|\xi|^2 + \int_t^T |Y_s^\varepsilon|^2 ds + \int_0^T |f(s, 0, 0)|^2 ds + \int_0^T |g(s, 0, 0)|^2 ds \right). \end{aligned}$$

Gronwall inequality and Burkholder-Davis-Gundy inequality show the desired result. \square

Lemma 3.3. Assume the assumptions of (H1)–(H4) hold. Then, there exists a constant $C_2 > 0$ such that

$$(i) \mathbb{E} \int_0^T \left(\frac{1}{\varepsilon} |D\varphi_\varepsilon(Y_s^\varepsilon)| \right)^2 ds \leq C_2;$$

$$(ii) \mathbb{E} \varphi(J_\varepsilon(Y_t^\varepsilon)) \leq C_2;$$

$$(iii) \mathbb{E} |Y_t^\varepsilon - J_\varepsilon(Y_t^\varepsilon)|^2 \leq \varepsilon^2 C_2.$$

Proof. (i) Given an equidistant partition of interval $[0, T]$ such that $0 = t_0 < t_1 < t_2 < \dots < t_n = T$ and $t_{i+1} - t_i = \frac{1}{n}$, the subdifferential inequality shows

$$\varphi_\varepsilon(Y_{t_{i+1}}^\varepsilon) \geq \varphi_\varepsilon(Y_{t_i}^\varepsilon) + (Y_{t_{i+1}}^\varepsilon - Y_{t_i}^\varepsilon) D\varphi_\varepsilon(Y_{t_i}^\varepsilon).$$

From (3.1), we obtain

$$\begin{aligned} \varphi_\varepsilon(Y_{t_i}^\varepsilon) + \frac{1}{\varepsilon} \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y_{t_i}^\varepsilon) D\varphi_\varepsilon(Y_s^\varepsilon) ds &\leq \varphi_\varepsilon(Y_{t_{i+1}}^\varepsilon) + \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y_{t_i}^\varepsilon) f(s, Y_s^\varepsilon, Z_s^\varepsilon) ds \\ &\quad + \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y_{t_i}^\varepsilon) g(s, Y_s^\varepsilon, Z_s^\varepsilon) dB_s \\ &\quad - 2 \sum_{j=1}^{\infty} \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y_{t_i}^\varepsilon) (Z_s^\varepsilon)^{(j)} dH_s^{(j)}. \end{aligned}$$

Summing up the above formula over i and letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} &\varphi_\varepsilon(Y_t^\varepsilon) + \frac{1}{\varepsilon} \int_0^T |D\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds \\ &\leq \varphi_\varepsilon(\xi) + \int_0^T D\varphi_\varepsilon(Y_s^\varepsilon) f(s, Y_s^\varepsilon, Z_s^\varepsilon) ds + \int_0^T D\varphi_\varepsilon(Y_s^\varepsilon) g(s, Y_s^\varepsilon, Z_s^\varepsilon) dB_s \\ &\quad - 2 \sum_{j=1}^{\infty} \int_0^T D\varphi_\varepsilon(Y_s^\varepsilon) (Z_s^\varepsilon)^{(j)} dH_s^{(j)}. \end{aligned}$$

Taking expectation on the both sides, we get

$$\mathbb{E} \varphi_\varepsilon(Y_t^\varepsilon) + \frac{1}{\varepsilon} \mathbb{E} \int_0^T |D\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds \leq \mathbb{E} \varphi_\varepsilon(\xi) + \mathbb{E} \int_0^T D\varphi_\varepsilon(Y_s^\varepsilon) f(s, Y_s^\varepsilon, Z_s^\varepsilon) ds. \quad (3.4)$$

For

$$\begin{aligned} D\varphi_\varepsilon(y) f(s, y, z) &\leq \frac{1}{2\varepsilon} |D\varphi_\varepsilon(y)|^2 + \frac{\varepsilon}{2} |f(s, y, z)|^2 \\ &\leq \frac{1}{2\varepsilon} |D\varphi_\varepsilon(y)|^2 + \varepsilon (|f(s, y, z) - f(s, 0, 0)|^2 + |f(s, 0, 0)|^2) \\ &\leq \frac{1}{2\varepsilon} |D\varphi_\varepsilon(y)|^2 + \varepsilon C |y|^2 + \varepsilon C \|z\|_{\ell^2}^2 + \varepsilon |f(s, 0, 0)|^2, \end{aligned}$$

the fact that $\varphi_\varepsilon(Y_t^\varepsilon) \geq 0$ and $\varphi_\varepsilon(\xi) \leq \varepsilon\varphi(\xi)$, we obtain

$$\frac{1}{2\varepsilon}\mathbb{E}\int_t^T |D\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds \leq C\mathbb{E}\left(\varphi(\xi) + \int_0^T |f(s, 0, 0)|^2 ds + T \sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2 + \int_0^T \|Z_t^\varepsilon\|_{\ell^2}^2 dt\right).$$

Lemma 3.2 shows the desired result.

(ii) From (3.4), we obtain

$$\mathbb{E}\varphi_\varepsilon(Y_t^\varepsilon) \leq \varepsilon\mathbb{E}\varphi(\xi) + \frac{1}{2\varepsilon}\mathbb{E}\int_t^T |D\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds + \varepsilon\mathbb{E}\int_t^T |f(s, Y_s^\varepsilon, Z_s^\varepsilon)|^2 ds.$$

Using $\varphi(J_\varepsilon(Y_t^\varepsilon)) \leq \frac{1}{\varepsilon}\varphi_\varepsilon(Y_t^\varepsilon)$ and (i), we obtain (ii).

The last part of the Lemma simply follows from the fact that

$$|x - J_\varepsilon(x)| = 2\varphi_\varepsilon(x) - 2\varepsilon\varphi(J_\varepsilon(x)).$$

□

In what follows, we aim to show that $(Y^\varepsilon, Z^\varepsilon)$ is a Cauchy sequence in $S^2 \times \mathcal{P}^2(l^2)$.

Lemma 3.4. *Assume the assumptions of (H1)–(H4) hold. Then, there exists a constant C_3 such that for all $\varepsilon, \delta > 0$*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t^\delta|^2 + \int_0^T \|Z_t^\varepsilon - Z_t^\delta\|_{\ell^2}^2 dt\right) \leq C_3(\varepsilon + \delta).$$

Proof. Applying the Itô formula to $|Y_t^\varepsilon - Y_t^\delta|^2$ yields that

$$\begin{aligned} |Y_t^\varepsilon - Y_t^\delta|^2 &= -2 \int_t^T (Y_s^\varepsilon - Y_s^\delta) \left(\frac{1}{\varepsilon} D\varphi_\varepsilon(Y_s^\varepsilon) ds - \frac{1}{\delta} D\varphi_\delta(Y_s^\delta) \right) ds \\ &\quad + 2 \int_t^T (Y_s^\varepsilon - Y_s^\delta) (f(s, Y_s^\varepsilon, Z_s^\varepsilon) - f(s, Y_s^\delta, Z_s^\delta)) ds \\ &\quad + 2 \int_t^T (Y_s^\varepsilon - Y_s^\delta) (g(s, Y_s^\varepsilon, Z_s^\varepsilon) - g(s, Y_s^\delta, Z_s^\delta)) dB_s \\ &\quad + \int_t^T |g(s, Y_s^\varepsilon, Z_s^\varepsilon) - g(s, Y_s^\delta, Z_s^\delta)|^2 ds - \int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 ds \\ &\quad - 2 \sum_{i=1}^\infty \int_t^T (Y_s^\varepsilon - Y_s^\delta) (Z_s^{\varepsilon, (i)} - Z_s^{\delta, (i)}) dH_s^{(i)}. \end{aligned} \tag{3.5}$$

Taking expectation, we obtain

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon - Y_t^\delta|^2 + \mathbb{E}\int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 ds &= -2\mathbb{E}\int_t^T (Y_s^\varepsilon - Y_s^\delta) \left(\frac{1}{\varepsilon} D\varphi_\varepsilon(Y_s^\varepsilon) ds - \frac{1}{\delta} D\varphi_\delta(Y_s^\delta) \right) ds \\ &\quad + 2\mathbb{E}\int_t^T (Y_s^\varepsilon - Y_s^\delta) (f(s, Y_s^\varepsilon, Z_s^\varepsilon) - f(s, Y_s^\delta, Z_s^\delta)) ds \\ &\quad + \mathbb{E}\int_t^T |g(s, Y_s^\varepsilon, Z_s^\varepsilon) - g(s, Y_s^\delta, Z_s^\delta)|^2 ds. \end{aligned} \tag{3.6}$$

Using the elementary inequality $2ab \leq \beta^2 a^2 + \frac{b^2}{\beta^2}$ for all $a, b \geq 0$ and (H2), we get

$$\begin{aligned} & (Y_s^\varepsilon - Y_s^\delta)(f(s, Y_s^\varepsilon, Z_s^\varepsilon) - f(s, Y_s^\delta, Z_s^\delta)) \\ & \leq \frac{2C}{1-\alpha} |Y_s^\varepsilon - Y_s^\delta|^2 + \frac{1-\alpha}{2} |Y_s^\varepsilon - Y_s^\delta|^2 + \frac{1-\alpha}{2} \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 \end{aligned}$$

and

$$|g(s, Y_s^\varepsilon, Z_s^\varepsilon) - g(s, Y_s^\delta, Z_s^\delta)|^2 \leq C |Y_s^\varepsilon - Y_s^\delta|^2 + \alpha \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2.$$

Noting (5) of Proposition 2.2, we obtain

$$\begin{aligned} & \mathbb{E} |Y_t^\varepsilon - Y_t^\delta|^2 + \frac{1-\alpha}{2} \mathbb{E} \int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 ds \\ & \leq C\varepsilon \mathbb{E} \int_t^T |Y_s^\varepsilon - Y_s^\delta|^2 ds \\ & \quad + 2 \left(\frac{1}{\varepsilon} + \frac{1}{\delta} \right) \mathbb{E} \int_t^T |D\varphi_\varepsilon(Y_s^\varepsilon)| |D\varphi_\delta(Y_s^\delta)| ds. \end{aligned} \tag{3.7}$$

Lemma 3.3 shows that

$$2 \left(\frac{1}{\varepsilon} + \frac{1}{\delta} \right) \mathbb{E} \int_t^T |D\varphi_\varepsilon(Y_s^\varepsilon)| |D\varphi_\delta(Y_s^\delta)| ds \leq (\varepsilon + \delta)C.$$

So, we can obtain

$$\mathbb{E} |Y_t^\varepsilon - Y_t^\delta|^2 + \mathbb{E} \int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 ds \leq C \mathbb{E} \int_t^T |Y_s^\varepsilon - Y_s^\delta|^2 ds + C(\varepsilon + \delta).$$

The Gronwall inequality shows that

$$\sup_{0 \leq t \leq T} \mathbb{E} |Y_t^\varepsilon - Y_t^\delta|^2 + \mathbb{E} \int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 ds \leq C(\varepsilon + \delta).$$

The Burkholder-Davis-Gundy inequality shows the desired result. \square

Proof of Theorem 3.1

Existence. Lemma 3.4 shows that $(Y^\varepsilon, Z^\varepsilon)$ is a Cauchy sequence in $S^2 \times \mathcal{P}^2(l^2)$. Denoting its limit by (Y, Z) , then it follows from Lemma 3.3 $(Y, Z) \in S^2 \times \mathcal{P}^2(l^2)$. For each $\varepsilon \geq 0$, define $U_t^\varepsilon = \frac{1}{\varepsilon} D\varphi_\varepsilon(Y_t^\varepsilon)$ and $\bar{U}_t^\varepsilon = \int_0^t U_s^\varepsilon ds$. Therefore (3.1) and Lemma 3.4 yield that for all $\varepsilon, \delta > 0$

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{U}_t^\varepsilon - \bar{U}_t^\delta|^2 \right) \leq C \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t^\delta|^2 + \int_0^T \|Z_t^\varepsilon - Z_t^\delta\|_{\ell^2}^2 dt \right),$$

which shows that (\bar{U}^ε) is a Cauchy sequence. Hence, there exists a measurable process \bar{U}_t such that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{U}_t^\varepsilon - \bar{U}_t|^2 \right) = 0.$$

Furthermore, Lemma 3.3 (i) shows that

$$\sup_{\varepsilon} \mathbb{E} \int_0^T |U_t^\varepsilon|^2 dt = \sup_{\varepsilon} \mathbb{E} \int_0^T \left(\frac{1}{\varepsilon} |D\varphi_\varepsilon(Y_t^\varepsilon)| \right)^2 dt < \infty,$$

which shows that \bar{U}_t^ε is bounded in the space $L^2(\Omega, H^1[0, T])$, and $(\bar{U}^\varepsilon)_\varepsilon$ converges weakly to a limit in that space and the limit is necessarily \bar{U} . In particular, \bar{U} is absolutely continuous. So, there exists a measurable process $(U_t)_{0 \leq t \leq T} \in \mathcal{H}^2$ such that $\bar{U}_t = \int_0^t U_s ds$.

Next, we show that $(Y_t, U_t) \in \partial\varphi, d\mathbb{P} \otimes dt$ -a.e. on $[0, T]$. Moreover, with the help of Lemma 5.8 in [14], and for all $0 \leq a < b \leq T, V \in \mathcal{H}^2([a, b])$, we obtain

$$\int_a^b U_t^\varepsilon (V_t - Y_t^\varepsilon) dt \rightarrow \int_a^b U_t (V_t - Y_t) dt, \text{ as } \varepsilon \rightarrow 0$$

in probability. In particular we have

$$\int_a^b U_t^\varepsilon (J_\varepsilon(Y_t^\varepsilon) - Y_t^\varepsilon) dt \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

which together with Proposition 2.2 provides that $U_t^\varepsilon \in \partial\varphi(J_\varepsilon(Y_t^\varepsilon))$ and

$$\int_a^b U_t^\varepsilon (V_t - J_\varepsilon(Y_t^\varepsilon)) dt + \int_a^b \varphi(J_\varepsilon(Y_t^\varepsilon)) dt \leq \int_a^b \varphi(V_t) dt.$$

Taking the \liminf in probability in the above inequality, we obtain

$$\int_a^b U_t (V_t - Y_t) dt + \int_a^b \varphi(Y_t) dt \leq \int_a^b \varphi(V_t) dt.$$

Since a, b and the process V are arbitrary, this shows that

$$U_t (V_t - Y_t) + \varphi(Y_t) \leq \varphi(V_t), d\mathbb{P} \otimes dt\text{-a.e. on } [0, T].$$

Taking limit on the both sides of (3.1), we obtain the existence of the solution.

Uniqueness. Let $(Y_t, U_t, Z_t)_{0 \leq t \leq T}$ and $(Y'_t, U'_t, Z'_t)_{0 \leq t \leq T}$ be two solutions of BDSDEs associated with (ξ, f, g, φ) . Define

$$(\Delta Y_t, \Delta U_t, \Delta Z_t)_{0 \leq t \leq T} = (Y_t - Y'_t, U_t - U'_t, Z_t - Z'_t)_{0 \leq t \leq T}.$$

Applying the Itô formula to $|\Delta Y_t|^2$ shows that

$$\begin{aligned} \mathbb{E} |\Delta Y_t|^2 + 2\mathbb{E} \int_t^T \Delta U_s \Delta Y_s ds + \mathbb{E} \int_t^T \|\Delta Z_s\|_{\ell^2}^2 ds \\ = 2\mathbb{E} \int_t^T \Delta Y_s [f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)] ds \\ + \mathbb{E} \int_t^T |g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)|^2 ds. \end{aligned} \quad (3.8)$$

Since $\partial\varphi$ is monotone, we obtain

$$\Delta U_s \Delta Y_s \geq 0, d\mathbb{P} \otimes dt\text{-a.e.}$$

Further, as the same procedure as Lemma 3.4, we obtain

$$\mathbb{E} |\Delta Y_t|^2 + \mathbb{E} \int_t^T \|\Delta Z_s\|_{\ell^2}^2 ds \leq C \mathbb{E} \int_t^T |\Delta Y_s|^2 ds + \frac{1}{2} \mathbb{E} \int_t^T \|\Delta Z_s\|_{\ell^2}^2 ds.$$

The Gronwall inequality shows the uniqueness of the solution.

4 Stochastic viscosity solutions of multivalued SPDEs

In this section, we derive the existence of the stochastic viscosity solution of a class of multivalued SPDE (1.2) via BDSDE with subdifferential operator and driven by Lévy process studied in the previous section.

4.1 Notion of stochastic viscosity solution of multivalued SPDEs

Let us recall $\mathbf{F}^B = \{\mathcal{F}_{t,T}^B\}_{0 \leq t \leq T}$ be the filtration generated by B . The object $\mathcal{M}_{0,T}^B$ denotes all the \mathbf{F}^B -stopping times τ such $0 \leq \tau \leq T$, a.s. and \mathcal{M}_∞^B is the set of all almost surely finite \mathbf{F}^B -stopping times. For generic Euclidean spaces E and E_1 , we state those spaces:

1. The symbol $C^{k,n}([0, T] \times E; E_1)$ stands for the space of all E_1 -valued functions defined on $[0, T] \times E$ which are k -times continuously differentiable in t and n -times continuously differentiable in x , and $C_b^{k,n}([0, T] \times E; E_1)$ denotes the subspace of $C^{k,n}([0, T] \times E; E_1)$ in which all functions have uniformly bounded partial derivatives.
2. For any sub- σ -field $\mathcal{G} \subseteq \mathcal{F}_T^B$, $C^{k,n}(\mathcal{G}, [0, T] \times E; E_1)$ (resp. $C_b^{k,n}(\mathcal{G}, [0, T] \times E; E_1)$) denotes the space of all $C^{k,n}([0, T] \times E; E_1)$ (resp. $C_b^{k,n}([0, T] \times E; E_1)$)-valued random variable that are $\mathcal{G} \otimes \mathcal{B}([0, T] \times E)$ -measurable;
3. $C^{k,n}(\mathbf{F}^B, [0, T] \times E; E_1)$ (resp. $C_b^{k,n}(\mathbf{F}^B, [0, T] \times E; E_1)$) is the space of all random fields $\varphi \in C^{k,n}(\mathcal{F}_T, [0, T] \times E; E_1)$ (resp. $C^{k,n}(\mathcal{F}_T, [0, T] \times E; E_1)$), such that for fixed $x \in E$ and $t \in [0, T]$, the mapping $\omega \rightarrow \alpha(t, \omega, x)$ is \mathbf{F}^B -progressively measurable.
4. For any sub- σ -field $\mathcal{G} \subseteq \mathcal{F}^B$ and a real number $p \geq 0$, $L^p(\mathcal{G}; E)$ denotes the set of all E -valued, \mathcal{G} -measurable random variable ξ such that $\mathbb{E}|\xi|^p < \infty$.

Furthermore, regardless of the dimension, we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the inner product and norm in E and E_1 , respectively. For $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, we denote $D_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$, $D_{xx} = (\partial_{x_i x_j}^2)_{i,j=1}^d$, $D_y = \frac{\partial}{\partial y}$, $D_t = \frac{\partial}{\partial t}$. The meaning of D_{xy} and D_{yy} is then self-explanatory. The coefficients

$$\begin{aligned} f &: \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \ell^2 \rightarrow \mathbb{R} \\ g &: \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \\ \sigma &: \mathbb{R}^d \rightarrow \mathbb{R}^d \\ u_0 &: \mathbb{R}^d \rightarrow \mathbb{R}, \end{aligned}$$

satisfying assumptions:

$$\begin{aligned} \text{(H5)} \quad & \begin{cases} |f(t, x, y, z)| \leq K(1 + |x| + |y| + \|z\|), \\ |u_0(x)| + |\varphi(u_0(x))| \leq K(1 + |x|). \end{cases} \\ \text{(H6)} \quad & \begin{cases} \|\sigma(x) - \sigma(x')\| \leq K|x - x'|, \\ |f(t, x, y, z) - f(t, x, y', z')| \leq K(|y - y'| + \|z - z'\|_{\ell^2}). \end{cases} \end{aligned}$$

(H7) The function $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R})$.

The definition of stochastic viscosity solution to MSPDIE (1.1) use the stochastic sub- and super-jets introduced by Buckdahn and Ma [9]. Let us recall the following needed definitions.

Definition 4.1. Let $\tau \in \mathcal{M}_{0,T}^B$, and $\xi \in \mathcal{F}_\tau$. We say that a sequence of random variables (τ_k, ξ_k) is a (τ, ξ) -approximating sequence if for all k , $(\tau_k, \xi_k) \in \mathcal{M}_\infty^B \times L^2(\mathcal{F}_\tau, \mathbb{R}^d)$ such that

- (i) $\xi_k \rightarrow \xi$ in probability;
- (ii) either $\tau_k \uparrow \tau$ a.s., and $\tau_k < \tau$ on the set $\{\tau > 0\}$; or $\tau_k \downarrow \tau$ a.s., and $\tau_k > \tau$ on the set $\{\tau < T\}$.

Definition 4.2. Let $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$ and $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$. We denote by $j_g^{1,2,+}u(\tau, \xi)$ the stochastic g -superjet of u at (τ, ξ) the set of $\mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(n)$ -valued and \mathcal{F}_τ^B -measurable random vector (a, p, X) ($\mathcal{S}(d)$ is the set of all symmetric $d \times d$ matrix) which is such that for all (τ, ξ) -approximating sequence (τ_k, ξ_k) , we have

$$\begin{aligned} u(\tau_k, \xi_k) &\leq u(\tau, \xi) + a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) + \frac{c}{2}(B_{\tau_k} - B_\tau)^2 + \langle p, \xi_k - \xi \rangle \\ &\quad + \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \frac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle \\ &\quad + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2). \end{aligned} \quad (4.1)$$

The \mathcal{F}_τ^B -measurable random vector (b, c, q) taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ is defined by

$$\begin{cases} b = g(\tau, \xi, u(\tau, \xi)), & c = (g \partial_u g)(\tau, \xi, u(\tau, \xi)) \\ q = \partial_x g(\tau, \xi, u(\tau, \xi)) + \partial_u g(\tau, \xi, u(\tau, \xi))p. \end{cases}$$

Similarly, $j_g^{1,2,-}u(\tau, \xi)$ denotes the set of all stochastic g -subjet of u at (τ, ξ) if the inequality in (4.1) is reversed.

Remark 4.3. Let us note that $\partial\varphi(y) = [\varphi'_l(y), \varphi'_r(y)]$, for every $y \in \text{Dom}(\varphi)$, where $\varphi'_l(y)$ and $\varphi'_r(y)$ denote the left and right derivatives of φ .

In order to simplify notation in the definition of the notion of stochastic viscosity solution of multivalued SPDIEs, we set

$$\begin{aligned} V_f(\tau, \xi, a, p, X) &= -a - \frac{1}{2} \text{Trace}(\sigma \sigma^*(\xi) X) - m_1 \langle p, \sigma(\xi) \rangle - \frac{1}{2} \int_{\mathbb{R}} \langle X \sigma(\xi), \sigma(\xi) \rangle y^2 \nu(dy) \\ &\quad - f\left(\tau, \xi, u(\tau, \xi), \int_{\mathbb{R}} \langle p, \sigma(\xi) y \rangle p_k(y) \nu(dy)\right). \end{aligned}$$

Definition 4.4. (1) A random field $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ which satisfies $u(T, x) = u_0(x)$, for all $x \in \mathbb{R}^d$, is called a stochastic viscosity subsolution of MSPDIE (1.1) if

$$u(\tau, \xi) \in \text{Dom}(\varphi), \quad \forall (\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d), \quad \mathbb{P}\text{-a.s.},$$

and at any $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$, for any $(a, p, X) \in \mathcal{J}_g^{1,2,+} u(\tau, \xi)$, it hold \mathbb{P} -a.s.

$$V_f(\tau, \xi, a, p, X) + \phi'_l(u(\tau, \xi) - \frac{1}{2}(g\partial_u g)(\tau, \xi, u(\tau, \xi))) \leq 0; \quad (4.2)$$

(2) A random field $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ which satisfies that $u(T, x) = u_0(x)$, for all $x \in \mathbb{R}^d$, is called a stochastic viscosity supersolution of MSPDIE (1.2) if

$$u(\tau, \xi) \in \text{Dom}(\phi), \quad \forall (\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d), \quad \mathbb{P}\text{-a.s.},$$

and at any $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$, for any $(a, p, X) \in \mathcal{J}_g^{1,2,-} u(\tau, \xi)$, it hold \mathbb{P} -a.s.

$$V(\tau, \xi, a, p, X) + \phi'_r(u(\tau, \xi) - \frac{1}{2}(g\partial_u g)(\tau, \xi, u(\tau, \xi))) \geq 0; \quad (4.3)$$

(3) A random field $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is called a stochastic viscosity solution of MSPDIE (1.1) if it is both a stochastic viscosity subsolution and a stochastic viscosity supersolution.

Remark 4.5. Observe that if f is deterministic and $g \equiv 0$, Definition 4.4 becomes the generalization of the definition of (deterministic) viscosity solution of MPDIE given by N'zi and Ouknine in [20].

To end this section, we state the notion of random viscosity solution which will be a bridge link to the stochastic viscosity solution and its deterministic counterpart.

Definition 4.6. A random field $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is called an ω -wise viscosity solution if for \mathbb{P} -almost all $\omega \in \Omega$, $u(\omega, \cdot, \cdot)$ is a (deterministic) viscosity solution of MSPDIE (1.2).

4.2 Doss-Sussmann transformation

In this section, using the Doss Sussmann transformation, our aim is to convert a multi-valued SPDIE to a PDIE with random coefficients so that the stochastic viscosity solution can be studied ω -wisely. We first establish the link between the g -super or sub jet of u the solution to multi-valued SPDIE and super or sub jet of v solution to the converter PDIE with random coefficients. For instance let us consider the stochastic flow $\eta \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d \times \mathbb{R})$, unique solution of the following stochastic differential equation in the Stratonovich sense:

$$\eta(t, x, y) = y + \int_t^T \langle g(s, x, \eta(s, x, y)), \circ dB_s \rangle, \quad (4.4)$$

where (4.4) should be viewed as going from T to t (i.e y should be understood as the initial value). Under the assumption (H7), the mapping $y \mapsto \eta(t, x, y)$ defines a diffeomorphism for all (t, x) , \mathbb{P} -a.s. such that its y -inverse $\varepsilon(t, x, y)$ is the solution to the following first-order SPDE:

$$\varepsilon(t, x, y) = y - \int_t^T \langle D_y \varepsilon(s, x, y), g(s, x, \eta(s, x, y)) \circ dB_s \rangle.$$

We refer the reader to their paper [10] for a lucid discussion on this topic. We have

Proposition 4.7. Assume that the assumptions (H1)–(H7) hold. If for $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$, $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ and (a_u, X_u, p_u) belongs to $\mathcal{J}_g^{1,2,+}u(\tau, \xi)$, then (a_v, X_v, p_v) belongs to $\mathcal{J}_0^{1,2,+}v(\tau, \xi)$, with $v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))$ and

$$\begin{cases} a_v = D_y \varepsilon(\tau, \xi, u(\tau, \xi)) a_u \\ p_v = D_y \varepsilon(\tau, \xi, u(\tau, \xi)) p_u + D_x \varepsilon(\tau, \xi, u(\tau, \xi)) \\ X_v = D_y \varepsilon(\tau, \xi, u(\tau, \xi)) X_u + 2D_{xy} \varepsilon(\tau, \xi, u(\tau, \xi)) p_u^* \\ \quad + D_{xx} \varepsilon(\tau, \xi, u(\tau, \xi)) + D_{yy} \varepsilon(\tau, \xi, u(\tau, \xi)) p_u p_u^*. \end{cases}$$

Conversely, if for $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$, $v \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ and $(a_v, X_v, p_v) \in \mathcal{J}_0^{1,2,+}v(\tau, \xi)$, then $(a_u, X_u, p_u) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$ with $u(\cdot, \cdot) = \eta(\cdot, \cdot, v(\cdot, \cdot))$ and

$$\begin{cases} a_u = D_y \eta(\tau, \xi, v(\tau, \xi)) a_v \\ p_u = D_y \eta(\tau, \xi, v(\tau, \xi)) p_v + D_x \eta(\tau, \xi, v(\tau, \xi)) \\ X_u = D_y \eta(\tau, \xi, v(\tau, \xi)) X_v + 2D_{xy} \eta(\tau, \xi, v(\tau, \xi)) p_v^* \\ \quad + D_{xx} \eta(\tau, \xi, v(\tau, \xi)) + D_{yy} \eta(\tau, \xi, v(\tau, \xi)) p_v p_v^*. \end{cases}$$

However, contrary to classical SPDE, the resulting PDIE from MSPDIE (1.2) due to Doss-Sussman transformation is not necessarily MPDIE studied by N'zi and Ouknine (see [20]). Therefore, we need the following version of viscosity solution for resulting PDIE obtain by Doss-Sussman transformation.

Corollary 4.8. Assume that the assumptions (H1)–(H7) hold. Let us define and consider $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$, $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$.

(1) for $(a_u, X_u, p_u) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$, u satisfies (4.2) if and only if $v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))$ satisfies

$$V_{\tilde{f}}(\tau, \xi, a_v, p_v, X_v) + \frac{\phi'_l(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \leq 0; \quad (4.5)$$

(2) for $(a_u, X_u, p_u) \in \mathcal{J}_g^{1,2,-}u(\tau, \xi)$, u satisfies (4.3) if and only if $v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))$ satisfies

$$V_{\tilde{f}}(\tau, \xi, a_v, p_v, X_v) + \frac{\phi'_l(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \geq 0; \quad (4.6)$$

where (a_v, p_v, X_v) is defined by Proposition 4.7 and

$$\begin{aligned} \tilde{f}(t, x, y, (\theta^k)_{k \geq 1}) &= \frac{1}{D_y \eta(t, x, y)} \left[f\left(t, x, \eta(t, x, y), D_y \eta(t, x, y) \theta^k + \eta_k^1(t, x, y)\right) \right. \\ &\quad - \frac{1}{2} (g \partial_u g)(t, x, \eta(t, x, y)) + L_x \eta(t, x, y) + \lambda \langle \sigma^*(x) D_{xy} \eta(t, x, y), \sigma(x) p_v \rangle \\ &\quad \left. + \frac{1}{2} \lambda D_{yy} \eta(t, x, y) |\sigma(x) p_v|^2 \right]. \end{aligned}$$

with $\theta^k = \int_{\mathbb{R}} \langle p_v, \sigma(x) u \rangle p_k(u) v(du)$ and $\lambda = 1 + \int_{\mathbb{R}} u^2 v(du)$.

Proof. Let $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$ be given and $(a_u, p_u, X_u) \in \mathcal{I}_g^{1,2,+} u(\tau, \xi)$. We assume that u is a stochastic subsolution of MSPDIE (1.2), i.e.

$$u(\tau, \xi) \in \text{Dom}(\varphi), \quad \forall (\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d), \quad \mathbb{P}\text{-a.s.},$$

such that

$$V_f(\tau, \xi, a, p, X) + \varphi'_l(u(\tau, \xi)) - \frac{1}{2}(g\partial_u g)(\tau, \xi, u(\tau, \xi)) \leq 0, \quad \mathbb{P}\text{-a.s.}$$

In view of Proposition 4.7 and since $D_y \eta(t, x, y) > 0$, for all (t, x, y) , we obtain by little calculation

$$V_{\tilde{f}}(\tau, \xi, a_v, p_v, X_v) + \frac{\varphi'_l(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \leq 0.$$

The converse part of (1) can be proved similarly. In the same manner one can show the second assertion (2). \square

5 Probabilistic representation result for stochastic viscosity solution to MSPDIEs

In this section, we aim to show that the solution of multivalued BDSDE with jump gives the viscosity solution of a semi-linear MSPDIE in the Markovian case.

5.1 A class of reflected diffusion process

We now introduce a class of diffusion process. Let $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a uniformly bounded function satisfying the uniform Lipschitz condition with some constant $C > 0$, for all $x, y \in \mathbb{R}^d$:

$$|\sigma(x) - \sigma(y)| \leq C|x - y|. \quad (5.1)$$

For each $(t, x) \in [0, T] \times \mathbb{R}^d$, from [17] and reference therein, let $\{X_s^{t,x}, s \in [t, T]\}$ be a unique pair of progressively measurable process, which is a solution to the following stochastic differential equation:

$$X_s^{t,x} = x + \int_t^s \sigma(X_r^{t,x}) dL_r. \quad (5.2)$$

Furthermore, we have the following proposition.

Proposition 5.1. *There exists a constant $C > 0$ such that for all $0 \leq t < t' \leq T$ and $x, x' \in \mathbb{R}^d$, such that*

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^4 \right] \leq C(|t' - t|^2 + |x - x'|^4)$$

5.2 Existence of viscosity solution for MSPDIEs

Fix $T > 0$ and for all $(t, x) \in [0, T] \times \mathbb{R}^d$, let $X_s^{t,x}$, $s \in [t, T]$ denote the solution of the SDE (5.2). And we suppose now that the data (ξ, f, g) of the multi-valued BDSDE with jump take the form

$$\begin{aligned}\xi &= u_0(X_T^{t,x}), \\ f(s, y, z) &= f(s, X_s^{t,x}, y, z), \\ g(s, y) &= f(s, X_s^{t,x}, y).\end{aligned}$$

And we give the following assumptions:

We assume that $u_0 \in C(\mathbb{R}^d; \mathbb{R})$, $f \in C([0, T] \times \mathbb{R}^d \times \mathbb{R} \times \ell^2; \mathbb{R})$ and $g \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R})$ such that assumptions (H1)–(H7) hold. It follows from the results of the Section 3 that, for all $(t, x) \in [0, T] \times \mathbb{R}^d$, there exists a unique triplet $(Y^{t,x}, Z^{t,x}, U^{t,x})$ for the solution of the following

$$\begin{aligned}(1) \quad & (Y_s^{t,x}, U_s^{t,x}) \in \partial\varphi, \quad d\mathbb{P} \otimes ds, \text{ -a.e. on } [t, T] \\ (2) \quad & Y_s^{t,x} + \int_s^T U_r^{t,x} dr = u_0(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}) dB_r \\ & \quad - \sum_{i=1}^{\infty} \int_s^T (Z_r^{t,x})_r^{(i)} dH_r^{(i)}, \quad t \leq s \leq T.\end{aligned}$$

We extend processes $Y^{t,x}, Z^{t,x}, U^{t,x}$ on $[0, T]$ by putting $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = 0$, $U_s^{t,x} = 0$, $s \in [0, t]$.

We have this result whose proof is similar to that of Theorem 2.1 appear in [25]

Proposition 5.2. *Let the ordered triplet $(Y_s^{t,x}, U_s^{t,x}, Z_s^{t,x})$ be the unique solution of the multi-valued BDSDE (5.2). Then, for $(s, t, x) \in [0, T] \times [0, T] \times \mathbb{R}^d$, the random field $(s, t, x) \mapsto \mathbb{E}'(Y_s^{t,x})$ is a.s. continuous ($Y^{t,x}$ has jump), where \mathbb{E}' is the expectation with respect to \mathbb{P}' , introduced at page 3.*

We are ready now to derive our main result in this section.

Theorem 5.3. *Assume the assumptions (H1)–(H7) be satisfied. Then, the function $u(t, x)$ defined by $u(t, x) = Y_t^{t,x} = \mathbb{E}'(Y_t^{t,x})$, does not depend on ω' , it follows from Proposition 5.2 that $u \in C(\mathcal{F}^B, [0, T] \times \mathbb{R}^d)$. Next, for all $(\tau, \xi) \in \mathcal{M}^B(0, T) \times L^2(\mathcal{F}^B, \mathbb{R}^d)$,*

$$\varphi(u(\tau(\omega), \xi(\omega))) = \varphi\left(Y_{\tau(\omega)}^{\tau(\omega), \xi(\omega)}\right) < \infty, \quad \mathbb{P}\text{-a.s.},$$

which implies that $u(\tau, \xi) \in \text{Dom}(\varphi)$ \mathbb{P} -a.s. Thus it remains to show that u is the stochastic viscosity solution to MSPDIE (1.2). In other word, using Corollary 4.8, it suffices to prove that $v(t, x) = \varepsilon(t, x, u(t, x))$ satisfies (4.5) and (4.6). In this fact, for each $(t, x) \in [0, T] \times \overline{\Theta}$, $\delta > 0$, let $\{(Y_s^{t,x,\delta}, Z_s^{t,x,\delta}), 0 \leq s \leq T\}$ denote the solution of the following BDSDE:

$$\begin{aligned}Y_s^{t,x,\delta} + \frac{1}{\delta} \int_s^T D\varphi_\delta(Y_r^{t,x,\delta}) dr &= u_0(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,\delta}, Z_r^{t,x,\delta}) dr \\ & \quad + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x,\delta}) dB_r \\ & \quad - \sum_{i=1}^{\infty} \int_s^T (Z_r^{t,x,\delta})_r^{(i)} dH_r^{(i)}.\end{aligned} \tag{5.3}$$

Setting $Y_t^{t,x,\delta} = u^\delta(t,x)$, it is shown by Theorem 3.6 in [2], that the function $v^\delta(t,x) = \varepsilon(t,x, u^\delta(t,x))$ is an ω -wise viscosity solution to this MSPDIE:

$$\begin{cases} (i) \left(\frac{\partial v^\delta}{\partial t}(t,x) - \left[\mathcal{L} v^\delta(t,x) + \tilde{f}_\delta(t,x, v^\delta(t,x), \sigma^*(x) \nabla v^\delta(t,x)) \right] \right) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ (ii) v(T,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (5.4)$$

where

$$\tilde{f}_\delta(t,x,y,z) = \tilde{f}(t,x,y,z) - \frac{\frac{1}{\delta} D\varphi_\delta(\eta(t,x,y))}{D_y \eta(t,x,y)}.$$

Moreover, letting Lemma 3.5 provide, along a subsequence that

$$|v^\delta(t,x) - v(t,x)| \rightarrow 0, \text{ a.s., as } \delta \rightarrow 0, \quad (5.5)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

On the other hand, for all $(\tau, \xi) \in \mathcal{M}^B(0,T) \times L^2(\mathcal{F}^B, \mathbb{R}^d)$ and $\omega \in \Omega$ be fixed, let consider $(a_v, p_v, X_v) \in \mathcal{J}_0^{1,2,+}(v(\tau(\omega), \xi(\omega)))$. Thus, since v^δ is an ω -wise viscosity solution to the MSPDIE (5.4), and by Crandall- Ishii-Lions in [12], there exist sequences

$$\begin{cases} \delta_n(\omega) \searrow 0, \\ (\tau_n(\omega), \xi_n(\omega)) \in [0,T] \times \mathbb{R}^d, \\ (a_v^n, p_v^n, X_v^n) \in \mathcal{J}_0^{1,2,+}(v^{\delta_n}(\tau_n(\omega), \xi_n(\omega))) \end{cases}$$

satisfying

$$\begin{aligned} (\tau_n(\omega), \xi_n(\omega), a_v^n, p_v^n, X_v^n, v^{\delta_n}(\tau_n(\omega), \xi_n(\omega))) &\rightarrow (\tau(\omega), \xi(\omega), a_v, p_v, X_v, v(\tau(\omega), \xi(\omega))) \\ n &\rightarrow \infty, \end{aligned}$$

such that for $(\tau_n(\omega), \xi_n(\omega)) \in [0,T] \times \mathbb{R}^d$,

$$V_{\tilde{f}(\omega)}(\tau_n(\omega), \xi_n(\omega), a_v^n, X_v^n, p_v^n) + \frac{\frac{1}{\delta_n} D\varphi_{\delta_n}(\eta(\tau_n(\omega), \xi_n(\omega), v^{\delta_n}(\tau_n(\omega), \xi_n(\omega))))}{D_y \eta(\tau_n(\omega), \xi_n(\omega), v^{\delta_n}(\tau_n(\omega), \xi_n(\omega)))} \leq 0. \quad (5.6)$$

In order to simplify the notation, we remove the dependence of ω . Let $y \in \text{Dom}(\varphi)$ such that $y < u(\tau, \xi) = \eta(\tau, \xi, v(\tau, \xi))$. The ucp convergence of v^{δ_n} to v implies that there exists $n_0 > 0$ such that $\forall n \geq n_0$, $y < \eta(\tau_n, \xi_n, v^{\delta_n}(\tau_n, \xi_n))$. Therefore, inequality (5.6) yields

$$\begin{aligned} & \left(\eta(\tau_n, \xi_n, v^{\delta_n}(\tau_n, \xi_n)) - y \right) V_{\tilde{f}_{\delta_n}}(\tau_n, \xi_n, a_v^n, X_v^n, p_v^n) \\ & \leq \left[\varphi(y) - \varphi(J_{\delta_n}(\eta(\tau, \xi, v^{\delta_n}(\tau, \xi)))) \right] \frac{1}{D_y \eta(\tau_n, \xi_n, v^{\delta_n}(\tau_n, \xi_n))}. \end{aligned}$$

Taking the limit in this last inequality, we get for all $y < \eta(\tau, \xi, v(\tau, \xi))$

$$V_{\tilde{f}}(\tau, \xi, a_v, X_v, p_v) \leq - \frac{\varphi(\eta(\tau, \xi, v(\tau, \xi))) - \varphi(y)}{\eta(\tau, \xi, v(\tau, \xi)) - y} \frac{1}{D_y \eta(\tau, \xi, v(\tau, \xi))},$$

which implies that

$$V_{\tilde{f}}(\tau, \xi, a_v, X_v, p_v) + \frac{\phi'_l(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \leq 0,$$

and derives that v satisfies (4.5). Hence, according to Corollary 4.8, u is a stochastic viscosity subsolution of MSPDIE (1.2). By similar arguments, one can prove that u is a stochastic viscosity supersolution of MSPDIE (1.2). This completes the proof. \square

References

- [1] A. Aman and N. Mrhardy, Obstacle problem for SPDE with nonlinear Neumann boundary condition via reflected generalized backward doubly SDEs, submitted (2010)
- [2] A. Aman and Y. Ren, Stochastic viscosity solution for stochastic PDIEs with nonlinear Neumann boundary condition, submitted (2010)
- [3] K. Bahlali, El. Essaky and Y. Ouknine, Reflected backward stochastic differnetial equations with jumps and locally Lipschitz coefficient, Random Oper. Stoch. Equ. 10 335–350 (2002)
- [4] K. Bahlali, El. Essaky and Y. Ouknine, Reflected backward stochastic differnetial equations with jumps and locally monotone coefficient, Stoch. Anal. Appl. 22 939–970 (2004)
- [5] V. Bally and A. Matoussi, Weak solutions for SPDEs and backward doubly SDEs, J. Theoret. Probab. 14 (2001) 125–164
- [6] B. Boufoussi, J-V. Casteren and N. Mrhardy, Generalized backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions, Bernoulli 13 423–446 (2007)
- [7] B. Boufoussi and N. Mrhardy, Multivalued stochastic partial differential equations via backward doubly stochastic differential equations, Stoch. Dyna. 8 271–294 (2008)
- [8] H. Brezis, Opérateurs maximaux monotones, Mathematics studies, North Holland, 1973
- [9] R. Buckdahn and J. Ma, Pathwise stochastic Taylor expansions and stochastic viscosity solutions for fully nonlinear stochastic PDEs, Ann. Probab. 30 1131–1171 (2002)
- [10] R. Buckdahn and J. Ma, Stochastic viscosity solutions for nonlinear stochastic partial differential equations (Part II), Stochastic Process. Appl. 93 205–228 (2001)
- [11] R. Buckdahn and J. Ma, Stochastic viscosity solutions for nonlinear stochastic partial differential equations (Part I), Stochastic Process. Appl. 93 181–204 (2001)
- [12] M. Crandall, H. Ishii and P.L. Lions, User’s guide to the viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 1–67 (1992)
- [13] C. Dellacheie and P.A. Meyer, Probabilites et Potentiel, Paris, Hermann, 1980
- [14] A. Gegout-Petit and E. Pardoux, Equations différentielles stochastiques rétrogrades réfléchies dans un convexe, Stochastics Stochastics Rep. 57 111–128 (1996)
- [15] G. Gong, An Introduction of stochastic differential equations, 2nd edition, Peking University of China, Peking, 2000
- [16] H. Kunita, Stochastic flows and stochastic differential equations, Cambridge University Press, 1990.

- [17] W. Laukajtys and L. Slominski, Penalization methods for reflecting stochastic differential equations with jumps, *Stochastics* **75** 275–293 (2003).
- [18] A. Matoussi and M. Scheutzow, Stochastic PDEs driven by nonlinear noise and backward doubly SDEs, *J. Theoret. Probab.* **15** 1–39 (2002)
- [19] M. N’zi and Y. Ouknine, Backward stochastic differential equations with jumps involving a subdifferential operator, *Random Oper. Stoch. Equ.* **8** 319–338 (2000)
- [20] M. N’zi and Y. Ouknine, Probabilistic interpretation for integral-partial with subdifferential operator, *Random Oper. Stoch. Equ.* **9** 87–101 (2000)
- [21] D. Nualart and W. Schoutens, Chaotic and predictable representation for Lévy processes, *Stochastic Process. Appl.* **90** 109–122 (2000)
- [22] D. Nualart and W. Schoutens, Backward stochastic differential equations and Feymann-Kac formula for Lévy processes, with applications in finance, *Bernoulli* **5** 761–776 (2001)
- [23] Y. Ouknine, Reflected BSDE with jumps, *Stochastics* **65** 111–125 (1998)
- [24] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation, *Systems Control Lett.* **14** 55–61 (1990)
- [25] E. Pardoux and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDE, *Probab Theory Related Fields* **88** 209–227 (1994)
- [26] E. Pardoux and A. Răşcanu, Backward stochastic differential equations with subdifferential operator and related variational inequalities, *Stochastic Process. Appl.* **76** (1998) 191–215
- [27] Y. Ren, Q. Zhou and A. Aman, Multivalued stochastic Dirichlet-Neumann problems and generalized backward doubly stochastic differential equations, submitted (2010)
- [28] Y. Ren and X. Fan, Reflected backward stochastic differential equations driven by a Lévy process, *ANZIAM J.* **50** 486–500 (2009)
- [29] Y. Ren, A. Lin and L. Hu, Stochastic PDIEs and backward doubly stochastic differential equations driven by Lévy processes, *J. Comput. Appl. Math.* **223** 901–907 (2009)
- [30] S. Tang and X. Li, Necessary condition for optional control of stochastic system with random jumps, *SIAM J. Control Optim.* **32** 1447–1475 (1994)
- [31] Q. Zhang and H. Zhao, Pathwise stationary solutions of stochastic partial differential equations and backward stochastic doubly stochastic differential equations on infinite horizon, *J. Funct. Anal.* **252** 171–219 (2007)